Practical Minimization of CVar-based Risk functions

Yassine Laguel

Univ. Grenoble Alpes, Grenoble, France

yassine.laguel@univ-grenoble-alpes.fr

Mots-clés : *Conditional Value at Risk, Risk Averse Optimization, Convex Optimization*

Risk measures play a major role in the design of safe and robust decisions for optimization problems under uncertainty. In this talk, we focus on convex risk functions involving the conditional value at risk. We present an open-source python toolbox with fast computational procedures.

1 Risk Averse Optimization

The conditional value at risk is an essential tool for the modeling of convex risk problems e.g. in mathematical finance or in machine learning. For a random variable X and a probability level $p \in [0, 1)$, the conditional value at risk of X, denoted $\mathrm{CVar}(X)$, is defined as :

$$\mathrm{CVar}(X) = \frac{1}{1-p} \int_{p'=p}^{1} Q_{p'}(X) dp' \tag{1}$$

where $Q_{p'}(X)$ denotes the p'-quantile of the random variable X. Minimizing CVar-based risk functions amounts to focus on the worst case scenarios. In this talk, we consider efficient first-order methods for the resolution of :

$$\min_{w \in \mathbb{R}^d} \mathrm{CVar}(f(w, \xi)) \tag{2}$$

where ξ is an $m-$dimensional random vector and $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is a designed loss.

Several standard risk-averse problems embrace this form. We mention for instance the so-called quantile regression [4], which can be reformulated as :

$$\min_{w \in \mathbb{R}^d} \mathrm{CVar}(Y - w^\top X) \tag{3}$$

where (X, Y) is the couple of random variables involved in the regression.

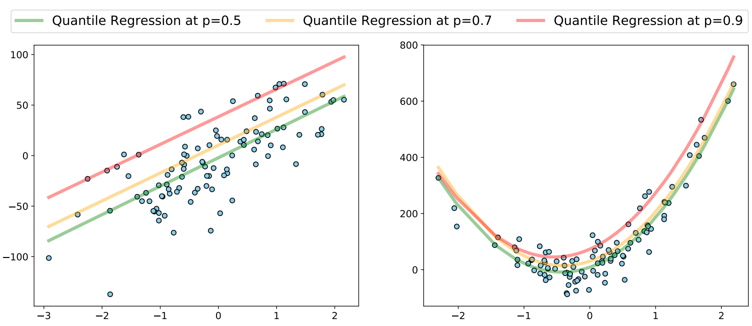

FIG. 1 – Quantile Regression for a linear and a quadratic model

2 Resolution by first-order methods : Approach and Toolbox

From an optimization perspective, (1) is usually reformulated as [4] :

$$\mathrm{CVar}(X) = \inf_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{1-p} \mathbb{E}[\max(X - \eta, 0)] \right\} \tag{4}$$

which enjoys some evident convex properties. However, the presence of a max term in (4) makes possibly non-smooth the objective function of (2). Under some assumptions of convexity of f, we propose, based on the theory of [5], to derive the subdifferential of CVar $\circ f$ at any point. The formula obtained enables us to locate the places where non-smoothness occurs and to link them to the properties of the random variable ξ involved.

We propose in addition to apply the standard smoothing procedure of [2] to obtain a smooth approximation of the conditional value at risk. This makes possible the use of BFGS algorithm. Computation of the associated gradient requires the resolution of an inner optimization problem for which we propose fast computational procedures.

We release an open-source python toolbox, `SPQR` [1], for the resolution of (2). It is built on top of `scikit-learn` [3], the popular ease-to-use machine learning library. The documentation is available at : `https://yassine-laguel.github.io/spqr/`.

Références

[1] Y. Laguel, J. Malick, and Z. Harchaoui. Superquantile minimization : Oracles and first-order algorithms. *Submitted*, 2019.

[2] Yu Nesterov. Smooth minimization of non-smooth functions. *Mathematical programming*, 103(1) :127–152, 2005.

[3] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay. Scikit-learn : Machine learning in Python. *Journal of Machine Learning Research*, 12 :2825–2830, 2011.

[4] R Tyrrell Rockafellar and Johannes O Royset. Superquantiles and their applications to risk, random variables, and regression. In *Theory Driven by Influential Applications*, pages 151–167. INFORMS, 2013.

[5] Andrzej Ruszczyński and Alexander Shapiro. Optimization of convex risk functions. *Mathematics of operations research*, 31(3) :433–452, 2006.