Computational Models for Cumulative Prospect Theory : Application to the Knapsack Problem Under Risk

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Introduction Cumulative Prospect Theory (CPT) is a well known model introduced by Kahneman and Tversky [5] in the context of decision making under risk to overcome some descriptive limitations of Expected Utility. In particular CPT makes it possible to account for the framing effect (outcomes are assessed positively or negatively relatively to a reference point) and the fact that people often exhibit different risk attitudes towards gains and losses.

We study here computational aspects related to the implementation of CPT for decision making in combinatorial domains. We consider the Knapsack Problem under Risk that consists of selecting the "best" subset of alternatives (investments, projects, candidates) subject to a budget constraint. The alternatives' outcomes may be positive or negative (gains or losses) and are uncertain due to the existence of several possible scenarios of known probability. Preferences over admissible subsets are based on the CPT model and we want to determine the CPT-optimal subset for a risk-averse Decision Maker (DM). The problem requires to optimize a non-linear function over a combinatorial domain.

We introduce two distinct computational models based on mixed-integer linear programming to solve the problem. These models are implemented and tested on randomly generated instances of different sizes to show the practical efficiency of the proposed approach. The complete version of this paper can be consulted in [2], which extends

The Cumultative Prospect Theory Let us consider a problem of decision making under risk with a finite set of states of nature $N = \{s_1, \dots, s_n\}$. The states represent possible scenarios under consideration, impacting differently the outcomes of the alternatives. Let p_i denote the probability of state s_i. Any feasible alternative is seen as an act in the sense of Savage. It is therefore characterized by a vector $x = (x_1, \dots, x_n)$ where $x_i \in \mathbb{R}$ denotes the outcome of x in state s_i. Let $x \in \mathbb{R}^n$ be the outcome vector such that $x_{(1)} \leq \dots \leq x_{(j-1)} < 0 \leq x_{(j)} \leq \dots \leq x_{(n)}$ with $j \in \{0, \dots, n\}$, the *Cumulative Prospect Theory* (CPT for short) is characterized by the following evaluation function :

$$g^u_{\varphi,\psi}(x) = \sum_{i=1}^{n} w_i u(x_i) \quad \textit{with} \quad w_i = \begin{cases} \varphi(\sum\limits_{k=i}^{n} p_{(k)}) - \varphi(\sum\limits_{k=i+1}^{n} p_{(k)}) & \textit{if} \quad (i) \geq (j) \\ \psi(\sum\limits_{k=1}^{i} p_{(k)}) - \psi(\sum\limits_{k=1}^{i-1} p_{(k)}) & \textit{if} \quad (i) < (j) \end{cases} \tag{1}$$

where φ and ψ are two real-valued increasing functions from $[0, 1]$ to $[0, 1]$ that assign 0 to 0 and 1 to 1, and u is a continuous and increasing real-valued utility function such that $u(0) = 0$ (hence $u(x)$ and x have the same sign).

In many situations decision makers are risk-averse. Roughly speaking, a strongly risk-avers decision maker tends to favor solution vectors having a more balanced profile among scenarios, which naturally reduces the risk associated to an optimal solution. Conditions for strong risk aversion to holds in CPT were first detailed in [4]. We will propose computational models to solve exactly the CPT-Knapsack problem assuming that the decision maker is strongly risk averse.

The CPT-Knapsack problem We present the CPT-Knapsack problem with m objects and n scenarios. The gain associated to an object j under a specific scenario i is denoted as v_{ij} and the weight associated to this object is w_j. We want to select the set of objects maximizing the CPT value without exceding the knapsack capacity C. The problem can be expressed with the following mathematical program :

$$(\mathcal{P}_1) \quad \text{s.t.} \begin{array}{l} \max \; g^u_{\varphi,\psi}(x_1, \ldots, x_n) \\ \left\{ \begin{array}{ll} x_i = \sum_{j=1}^m u(v_{ij}) y_j & i = 1, \ldots, n \\ \sum_{j=1}^m y_j \leq C \end{array} \right. \\ y_j \in \{0, 1\}, \; j = 1, \ldots, m \end{array}$$

where y_j is the decision variable relative to the selection of candidate j, $j = 1, \ldots, m$. The non-linearity of the CPT aggregation function makes it difficult to propose mathematical program to solve this problem. Using conditions of strong risk aversion, we will propose two linearization of the former mathematical program, under the assumption that $u(x) = x$.

Resolution approach Using theory of capacities and the notion of core, we propose a reformulation of the CPT aggregation function and use it to linearize program $\mathcal{P}_1$ in a new linear program $\mathcal{P}_2$. This work extends the linearization of a concave Choquet Integral proposed in [1]. Program $\mathcal{P}_2$ has an exponential number of continuous variables (depending on the number of scenarios), a polynomial number of binary variables (also depending on n) and a polynomial number of constraints. Therefore, it is able to solve instances with a large number of objects but a limited number of scenarios (less than 7). To overcome this limitation, we propose a new formulation of the CPT aggregation function, under the assumption that φ and ψ are two piecewise-linear functions. We propose then a new mixed integer program $\mathcal{P}_3$, with a polynomial number of variables and constraints. This program is then able to solve faster instances with a larger number of objects and scenarios than $\mathcal{P}_2$. Program $\mathcal{P}_3$ extends the linearization of the Weighted OWA operator, first proposed in [3].

Références

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