Comparison of symmetry-breaking techniques for structured (sub-)symmetries in Integer Linear Programming

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Symmetries arising in integer linear programs can impair the solution process, in particular when symmetric solutions lead to an excessively large Branch and Bound (B&B) search tree. Various techniques, so called *symmetry-breaking techniques*, are available to handle symmetries in integer linear programs of the form $(ILP)\ \ \min\{cx \mid x \in \mathcal{X}\}$, with $\mathcal{X} \subseteq \mathcal{P}(m,n)$ where $\mathcal{P}(m,n)$ is the set of $m \times n$ binary matrices. A symmetry is defined as a permutation π of the indices $\{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ such that for any solution $x \in \mathcal{X}$, $\pi(x)$ is also solution with the same cost, *i.e.*, $\pi(x) \in \mathcal{X}$ and $c(x) = c(\pi(x))$. The *symmetry group* $\mathcal{G}$ of (ILP) is the set of all such permutations. It partitions the solution set $\mathcal{X}$ into *orbits*, *i.e.*, two matrices are in the same orbit if there exists a permutation in $\mathcal{G}$ sending one to the other. A *subproblem* is problem (ILP) restricted to a subset of $\mathcal{X}$. In [2], symmetries arising in solution subsets of (ILP) are called *sub-symmetries*. Such sub-symmetries may not exist in $\mathcal{G}$.

In this article, we focus on structured symmetries arising from (sub-)symmetry groups containing all sub-column permutations of a given solution submatrix. These symmetry groups are assumed to be known or previously detected [4].

A first idea to break symmetries is to reformulate the problem using integer variables summing the variables along orbits. Such a reformulation aggregates variables, thus reducing the size of the resulting ILP [10]. However, it can be used only when aggregated solutions can be disaggregated. This is for example the case when the integer decomposition property [1] holds.

A more general idea to break symmetries is, in each orbit to pick one solution, defined as the *representative*, and then restrict the solution set to the set of all representatives. The most common choice of representative is based on the lexicographical order. Column $y \in \{0,1\}^m$ is said to be *lexicographically greater than* column $z \in \{0,1\}^m$ if there exists $i \in \{1, ..., m-1\}$ such that $\forall i' \leq i$, $y_{i'} = z_{i'}$ and $y_{i+1} > z_{i+1}$, *i.e.*, $y_{i+1} = 1$ and $z_{i+1} = 0$. We write $y \succ z$ (resp. $y \succeq z$) if y is lexicographically greater than z (resp. greater than or equal to z). A technique is said to be *full symmetry-breaking* (resp. *partial symmetry-breaking*) if the solution set is exactly (resp. partially) restricted to the representative set. A symmetry-breaking technique is said to be *flexible* if, at any node of the B&B tree, the branching rule can be derived from any linear inequality on the variables.

Most techniques based on branching and pruning rules [13, 8, 2] are either full symmetry-breaking or flexible. Variable fixing [8, 2] is both full symmetry-breaking and flexible. Other symmetry-breaking techniques rely on full or partial symmetry-breaking inequalities. Such techniques are flexible. Note that the size of the LP solved at each node of the branching tree is generally invariant under pruning and variable fixing techniques, whereas it is increased by the use of symmetry-breaking inequalities. Symmetry-breaking inequalities can be derived from the linear description of the convex hull of an arbitrary representative set [6]. In most works, each chosen representative x is lexicographically maximal in its orbit, *i.e.*, $x \succeq g(x)$, for each $g \in \mathcal{G}$. The convex hull of the latter representative set is called the *symmetry-breaking polytope* [6]. When solution x is a matrix and when the symmetry group $\mathcal{G}$ acts on the columns of x, the symmetry-breaking polytope is called *orbitope*. Even if complete linear descriptions for orbitopes may be hard to reach in general, integer programming formulations for these polytopes still yield full symmetry-breaking inequalities [12]. When symmetry group $\mathcal{G}$ is the *symmetric group* $\mathfrak{S}_n$ acting on the columns of x, *i.e.*, the set containing all column permutations, then the chosen representative x of an orbit may be such that its columns $x(1)$, ..., $x(n)$ are lexicographically non-increasing, *i.e.*, for all $j < n$, $x(j) \succeq x(j+1)$. The convex hull of all $m \times n$ binary matrices with lexicographically non-increasing columns is called the *full orbitope* [9]. Sub-symmetries and sub-orbitopes are introduced in [2] as a generalization of symmetries and full orbitopes to a given set of solution subsets. Another class of symmetry-breaking inequalities aims to ensure that the integer solutions lie in the full orbitope. Friedman inequalities, introduced in [5], are full symmetry-breaking, but feature 2^{m-i} coefficients that might cause numerical intractability. Alternative inequalities defined in [9, 13] as *column inequalities* feature ternary coefficients at the expense of losing the full symmetry-breaking property. A partial form of Friedman inequalities with binary coefficients is proposed in [7, 11], in order to enforce that the total number of ones in each column is non-increasing. The authors of [6] propose inequalities with $\{-1, 0, 1\}$ coefficients, ensuring that any integer point is in the full orbitope. These inequalities are therefore full-symmetry-breaking. They are in exponential number, but can be separated in linear time. In [3], the authors define full symmetry-breaking inequalities handling sub-symmetries arising from solution subsets whose symmetry groups contain the symmetric group acting on some sub-columns of x are assumed to be known. One additional variable per subset Q considered may be needed in these inequalities, depending on whether variables x are sufficient to indicate that "x belongs to subset Q".

In this article, we propose an experimental comparison of various symmetry-breaking techniques on instances of a variant of the Unit Commitment Problem (UCP). The techniques considered are variable aggregation, modified orbital branching, orbitopal fixing for the full orbitope, column inequalities, partial Friedman inequalities, and two families of sub-symmetry-breaking inequalities. The considered variant of the UCP is with constraints on the minimum up and down times of each unit. This variant is called the Min-up/min-down Unit Commitment Problem (MUCP) as defined in [14]. When the MUCP is considered, the integer decomposition property holds for the classical formulation and thus efficient aggregation techniques apply [10]. A variant of the MUCP is with constraints limiting power variations, referred to as *ramp constraints*. When the ramp-constrained MUCP is considered, the integer decomposition property does not hold anymore for the classical formulation, then the corresponding aggregated solutions can no longer be disaggregated. We show that the sub-symmetry-breaking inequalities from [3] outperform other state-of-the-art symmetry-breaking techniques in this setting.

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