# Estimation in Periodic Restricted EXPAR(p) models by Conditional Least Squares method

BECILA Sabah[1], MERZOUGUI Mouna[2]

[1] Université Badji Mokhtar Annaba, Lab-LaPS, Annaba, Algérie
{sabahbc@gmail.com
[2] Université Badji Mokhtar Annaba, Lab-LaPS, Annaba, Algérie
merzouguimouna@yahoo.fr

**Mots-clés** : *Nonlinear time series, Nonlinear periodic stochastic difference equation, Periodic restricted exponential autoregressive model, Least squares estimation, Asymptotic properties.*

## 1 Introduction

Periodic time series models have been extensively used in the recent decades to describe many series with periodic dynamics. The inability of SARIMA models to adequately represent many seasonal time series exhibiting a periodic autocovariance structure has motivated the research in the periodically correlated processes. This notion, introduced by Gladyshev (1960), was exploited in a variety of new classes of time series models, among them, the periodic GARCH (Bollerslev and Ghysels (1996)), the periodic bilinear (Bibi and Gautier (2005)) and the mixture periodic autoregressive model (Shao (2006)).
In this paper, we extend the class of periodic restricted exponential autoregressive model (PEXPAR(1)) discussed in Merzougui et al. (2016) to order p. PEXPAR series satisfy a nonlinear difference equation similar to that for EXPAR models with parameters and white noise variances which change periodically with season.
The class of exponential autoregressive (EXPAR) models introduced by Ozaki (1980) and Haggan and Ozaki (1981) has shown their appropriateness in capturing certain well-known features of nonlinear vibration theory, such as amplitude dependent frequency, jump phenomena and limit cycle behavior, these models are autoregressive in form with amplitude dependent exponential coefficients.
This paper deals with the least squares estimation of the periodic restricted EXPAR(p) model.

## 2 Main results

### 2.1 Periodic restricted EXPAR model

The proces $\{Y_t; t \in \mathbb{Z}\}$ is said to follow a Periodic Restricted Exponential Autoregressive $PEXPAR_S(p_t)$, with period $S(S \geq 2)$ , if it is a solution of a nonlinear periodic stochastic difference equation of the form :

$$Y_t = \sum_{j=1}^{p_t}(\varphi_{t,j} + \pi_{t,j}exp(-\gamma Y_{t-1}^2))Y_{t-j} + \varepsilon_t, t \in \mathbb{Z} \tag{1}$$

Where $\{\varepsilon_t; t \in \mathbb{Z}\}$ is i.i.d. process with continuous density $f_{\sigma_t}(.)$, not necessarily Gaussian, with mean 0 and finite variance $\sigma_t^2$. The autoregressive parameters $\varphi_{t,j}, \pi_{t,j}, \forall t \in \mathbb{Z}$ and $j = 1, \cdots, p$, the order $p_t$ and the innovation variance $\sigma_t^2$ are periodic, in time, with period S, i.e.,
$\varphi_{t+kS,j} = \varphi_{t,j}, \pi_{t+kS,j} = \pi_{t,j}, p_{t+kS} = p_t$, and $\sigma_{t+kS}^2 = \sigma_t^2, \forall k, t \in \mathbb{Z}$ and $j = 1, \cdots, p_t$.

The nonlinear parameter,$\gamma > 0$, is known. A heuristic determination of $\gamma$ from data is $\widehat{\gamma} = -\frac{log\epsilon}{maxY_t^2}$, where $1 \leq t \leq n$ and $\epsilon$ is a small number (cf. Shi et al.(2001)).

Putting $t = i + S\tau, i = 1, 2, \ldots, S$ and $\tau \in \mathbb{Z}$ and taking $p = maxp_i, i \in \{ 1, 2, \cdots, S\}$. where $\varphi_{i,j} = 0, \pi_{i,j} = 0$, for each $j > p_i$, one can rewrite equation (1) in the equivalent form :

$$Y_{i+S\tau} = \sum_{j=1}^{p}(\varphi_{i,j} + \pi_{i,j}exp(-\gamma Y^2_{i+S\tau-1}))Y_{i+S\tau-j} + \varepsilon_{i+S\tau}, i = 1, \ldots, S, \tau \in \mathbb{Z} \qquad (2)$$

Let
$\underline{\varphi}_i = (\varphi_{i,1}, \pi_{i,1}, \ldots, \varphi_{i,p}, \pi_{i,p})', i = 1, \ldots, S$ and $\underline{\varphi} = (\underline{\varphi}'_1, \ldots, \underline{\varphi}'_S)' \in \mathbb{R}^{2pS}$.

We make the following assumptions :

**A1** : The periodic exponential autoregressive parameters $\underline{\varphi}$ satisfy the strict stationarity periodically condition of (2). A sufficient condition is : All the roots of associated characteristic equation

$$z^p - c_{i,1}z^{p-1} \cdots - c_{i,p} = 0$$

are inside the unit circle, where $c_{i,j} = max\{|\varphi_{i,j}|, |\varphi_{i,j} + \pi_{i,j}|\}, j = 1, \cdots, p; i = 1, \cdots, S.$

**A2** : The periodically ergodic process $\{Y_t; t \in \mathbb{Z}\}$ is such that $E(Y_t^4) < \infty$, for any $t \in \mathbb{Z}$.

### 2.2 Parameter estimation

The estimation of the parameters $\underline{\varphi}$ of the model(2) is a linear optimisation problem, we can solve it using the least squares procedure. Suppose that we have observations $\{Y_1, \cdots, Y_N\}$ from (2), $N = mS$, and define the conditional sum of squares

$L_N(\underline{\varphi}) = \sum_{i=1}^{S} L_{i,m}(\underline{\varphi})$

$L_N(\underline{\varphi}) = \sum_{i=1}^{S}(\sum_{\tau=r+1}^{m-1}(Y_{S\tau+i} - E_{\underline{\varphi}}(Y_{S\tau+i}|\beta_{S\tau+i-1}))^2)$

$L_N(\underline{\varphi}) = \sum_{i=1}^{S}(\sum_{\tau=r+1}^{m-1}(Y_{S\tau+i} - \sum_{j=1}^{p}(\varphi_{i+j} + \pi_{i+j}exp(-\gamma Y^2_{S\tau+i-1}))Y_{S\tau+i-j})^2).$

where $r = \left[\frac{p}{S}\right]$, with $[x]$ denotes the integer part of $x$, $\beta_{S\tau+i-1}$ is the $\sigma$-algebra generated by the past of the process up to time $S\tau+i-1$ and $E_{\underline{\varphi}}(.|.)$ is the conditional expectation assuming that $\underline{\varphi}$ is the true parameter.

The estimate $\widehat{\underline{\varphi}_i} = (\widehat{\varphi}_{i,1}, \widehat{\pi}_{i,1}, \cdots, \widehat{\varphi}_{i,p}, \widehat{\pi}_{i,p})'$, for a fixed season $i$, is a solution to the estimating equations

$$\frac{\partial L_{i,m}(\varphi)}{\partial \varphi_{i,j}} = 0 \text{ and } \frac{\partial L_{i,m}(\varphi)}{\partial \pi_{i,j}} = 0, \; j = 1, \ldots, p.$$

The solution for a fixed season $i$ is

$$\widehat{\underline{\varphi}_i} = \begin{pmatrix} M_{i,1,1} & \cdots & M_{i,1,p} \\ \vdots & \ddots & \vdots \\ M_{i,p,1} & \cdots & M_{i,p,p} \end{pmatrix}^{-1} \times \begin{bmatrix} \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-1}Y_{S\tau+i} \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-1}Y_{S\tau+i}exp(-\gamma Y^2_{S\tau+i-1}) \\ \vdots \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-p}Y_{S\tau+i} \\ \sum_{\tau=r+1}^{m-1} Y_{S\tau+i-p}Y_{S\tau+i}exp(-\gamma Y^2_{S\tau+i-1}) \end{bmatrix}, \qquad (3)$$

$$\widehat{\sigma}_i^2 = \frac{1}{m-r-1} \sum_{\tau=r+1}^{m-1} (Y_{S\tau+i} - \sum_{j=2}^{p} (\widehat{\varphi}_{i,j} + \widehat{\pi}_{i,j} exp(-\gamma Y_{S\tau+i-1}^2)) Y_{S\tau+i-j})^2, \tag{4}$$

Where for $j, k = 1, \cdots, p$,

$$Mi,j,k = \begin{pmatrix} \sum\limits_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} & \sum\limits_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} exp(-\gamma Y_{S\tau+i-1}^2) \\ \sum\limits_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} exp(-\gamma Y_{S\tau+i-1}^2) & \sum\limits_{\tau=r+1}^{m-1} Y_{S\tau+i-j} Y_{S\tau+i-k} exp(-2\gamma Y_{S\tau+i-1}^2) \end{pmatrix}.$$

**Remark 1.** *For p=1, we obtain the estimates of the periodic restricted EXPAR(1) model (cf. Merzougui, 2017).*

**Theorem 1.** *Suppose that $\{Y_t\}$, satisfying (2), is strictly stationary, then the least squares estimators (3) and (4) are strongly consistent as $m \to \infty$. That is*

$$\underline{\widehat{\varphi}}_i \xrightarrow{a.s} \underline{\varphi}_i \text{ and } \widehat{\sigma}_i^2 \xrightarrow{a.s} \sigma_i^2,$$

*and we have*

$$\sqrt{m}(\underline{\widehat{\varphi}}_i) - \underline{\varphi}_i) \underset{m \to \infty}{\overset{D}{\to}} N(\underline{0}_{2p}, \sigma_i^2 \Gamma_i^{-1}),$$

*where*

$$\Gamma_i = \begin{pmatrix} \Gamma_{i,1,1} & \cdots & \Gamma_{i,1,p} \\ \vdots & \ddots & \vdots \\ \Gamma_{i,p,1} & \cdots & \Gamma_{i,p,p} \end{pmatrix},$$

*and*

$$\Gamma_{i,j,k} = \begin{pmatrix} E(Y_{i-j} Y_{i-k}) & E(Y_{i-j} Y_{i-k} exp(-\gamma Y_{i-1}^2)) \\ E(Y_{i-j} Y_{i-k} exp(-\gamma Y_{i-1}^2)) & E(Y_{i-j} Y_{i-k} exp(-2\gamma Y_{i-1}^2)) \end{pmatrix}, j, k = 1, \cdots, p.$$

### 2.3 Simulation results

The performance of the estimation is shown via small simulation. The restricted $PEXPAR_2(2)$ model is used to generate time series for sizes n=200,400,800. We consider 1000 Monte Carlo replications and report the LS estimations, their bias and their standard deviations. The table 1 gives the estimation with the parameters $\underline{\varphi} = (0.6, -1, 0.3, -0.5; -0.5, 1, -0.4, 0.8)', \gamma = 1$ and normal white noise $\underline{\sigma^2} = (0.6, 1)'$. The choice of the values of the parameters was taken such that the model fulfill the condition **A1**, see Figure 1. The box plots and the Q-Q plots of the errors are given in Figure 2 and 3, respectively.
Table 1 show that the estimates are close to the true values and the standard deviation decreases when n becomes larger and we remark that the standard deviation of $\varphi_{i,j}$ are smaller than those of $\pi_{i,j}$. This is confirmed by the box plots where we observe that the errors are more consistent for $\varphi_{i,j}$ and the range is larger for $\pi_{i,j}$ ,but in all cases the errors are centered on 0. On the other hand, the Q-Q plots show that the errors are normal.

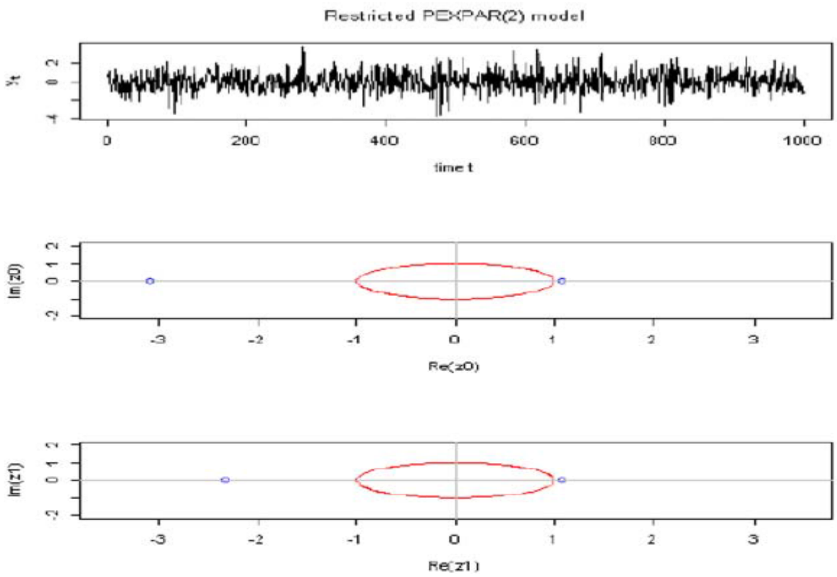


**Figure 1.** Simulated series and inverse roots of the characteristic equation of restricted $PEXPAR_2(2)$ and $n = 1000$.

| | $\widehat{\varphi}_{1,1}$ | $\widehat{\pi}_{1,1}$ | $\widehat{\varphi}_{1,2}$ | $\widehat{\pi}_{1,2}$ | $\widehat{\varphi}_{2,1}$ | $\widehat{\pi}_{2,1}$ | $\widehat{\varphi}_{2,2}$ | $\widehat{\pi}_{2,2}$ |
|---|---|---|---|---|---|---|---|---|
| $n = 200$ | 0.5969 | − 0.9943 | 0.3003 | − 0.4951 | − 0.5077 | 1.0083 | − 0.3760 | 0.7644 |
| bias | − 0.0030 | 0.0056 | 0.0003 | 0.0048 | − 0.0077 | 0.0083 | 0.0239 | − 0.0355 |
| sd | 0.0954 | 0.3473 | 0.1486 | 0.2513 | 0.2003 | 0.5182 | 0.1999 | 0.3008 |
| $n = 400$ | 0.5996 | − 0.9969 | 0.3055 | − 0.5030 | − 0.5013 | 1.0020 | − 0.3901 | 0.7819 |
| bias | − 0.0003 | 0.0030 | 0.0055 | − 0.0030 | − 0.0013 | 0.0020 | 0.0098 | − 0.0180 |
| sd | 0.0654 | 0.2357 | 0.1028 | 0.1778 | 0.1381 | 0.3492 | 0.1354 | 0.2031 |
| $n = 800$ | 0.5988 | − 0.9893 | 0.3023 | − 0.5049 | − 0.5030 | 1.0045 | − 0.3950 | 0.7894 |
| bias | − 0.0011 | 0.0106 | 0.0023 | − 0.0049 | − 0.0030 | 0.0045 | 0.0049 | − 0.0105 |
| sd | 0.0461 | 0.1643 | 0.0723 | 0.1226 | 0.0979 | 0.2463 | 0.0959 | 0.1443 |

TAB. 1 – Estimation results for restricted $PEXPAR_2(2)$

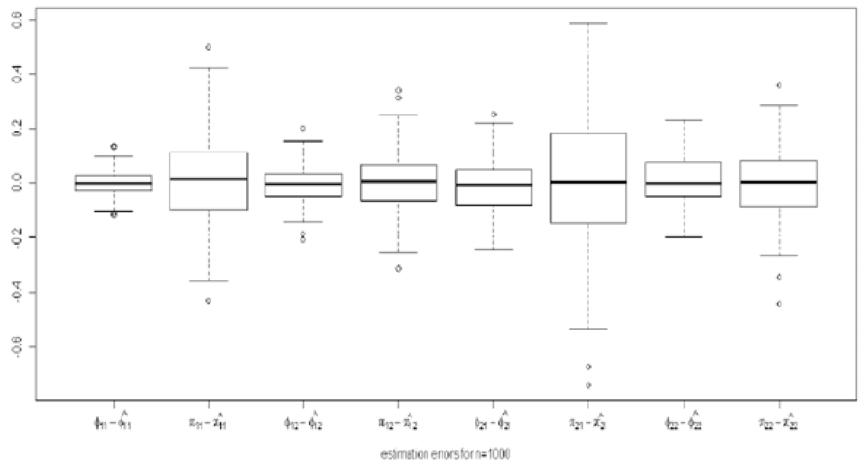


**Figure 2.** Box plots of the errors from estimates of 200 replications of restricted $PEXPAR_2(2)$ and $n = 1000$.

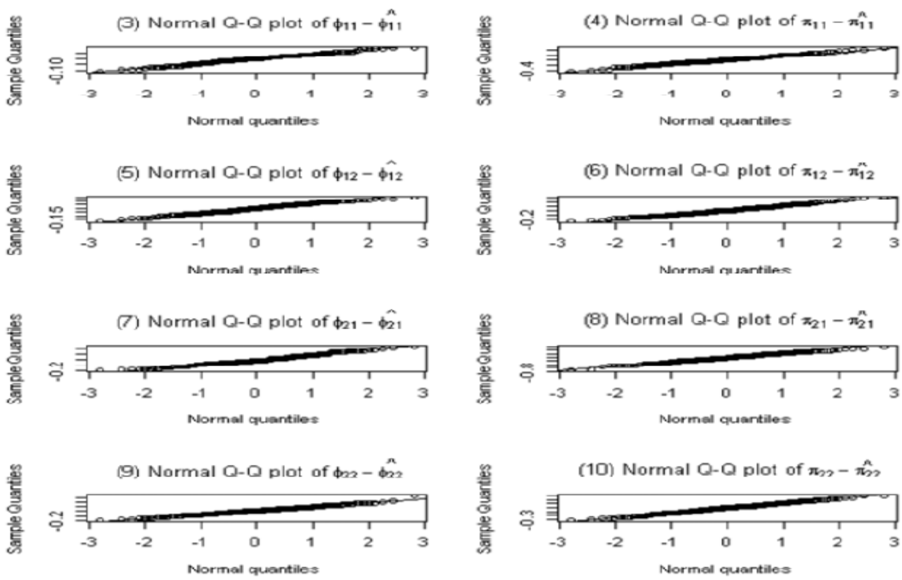


**Figure 3.** The Q-Q plots of the errors from estimates of 200 replications of restricted $PEXPAR_2(2)$ and $n = 1000$.

# 3 Conclusion

In this study, we have used the linear least squares method for the estimation of the periodic restricted EXPAR(p) model, consistency and asymptotic normality are derived and simulated series checked the asymptotic properties. This LS estimator can be used as an initial estimator in adaptive estimation.
As a part of future research, the authors study the Nonlinear LS and Quasi ML estimation of the periodic (unrestricted) EXPAR(p) model.
We have considered, here, a sufficient condition of strict stationarity but this subject merit further research.

# Références


[1] Haggan, V. and Ozaki,T. (1981), Modeling nonlinear random vibrations using an amplitude-dependent autoregressive time series model. Biometrik. 68(1), 96-189.

[2] Chan, K. S. and Tong, H. (1985). On the use of the deterministic Lyapunov function for the ergodicity of stochastic dierence equations. Advances in applied probability, 17(3), 666-678.

[3] Koul, H. L. and Schick, A.(1997). E cient estimation in nonlinear autoregressive time series models. Bernoulli, 3, 247-277.

[4] Merzougui, M. (2017) : Estimation in periodic restricted EXPAR(1) models. Communications in Statistics - Simulation and Computation, D.O.I. 10.1080/03610918.2017.1361975.

[5] Merzougui, M., Dridi, H. and Chadli, A. (2016) : Test for Periodicity in restrictive EXPAR Models. Communications in Statistics Theory and Methods, 45(9), 2770-2783.